

Classical relativistic ideal gas in thermodynamic equilibrium in a uniformly accelerated reference frame

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Abstract

A classical (non-quantum-mechanical) relativistic ideal gas in thermodynamic equilibrium in a uniformly accelerated frame of reference is studied using Gibbs's micro-canonical and grand canonical formulations of statistical mechanics. Using these methods explicit expressions for the particle, energy and entropy density distributions are obtained, which are found to be in agreement with the well known results of the relativistic formulation of Boltzmann's kinetic theory. Explicit expressions for the total entropy, total energy and rest mass of the gas are obtained. The position of the center of mass of the gas in equilibrium is found. The non-relativistic and ultrarelativistic approximations are also considered. The phase space volume of the system is calculated explicitly in the ultrarelativistic approximation.

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Introduction:

The relativistic ideal gas in a gravitational field has been extensively studied using the relativistic Boltzmann equation [1, 2, 3, 4, 5, 6, 7]. This theory has found interesting applications in astrophysics, cosmology and nuclear physics [6, 7].

General relativity is a local theory. The relativistic Boltzmann distribution function too is defined locally. It is therefore not surprising that Boltzmann's kinetic theory has been the method employed in the study of relativistic gases in gravitational fields.

Gibbs's microcanonical formulation of statistical mechanics deals with systems of finite volumes and finite energies. To our knowledge, the methods of statistical mechanics, which are based on the microcanonical formulation, have not been employed so far to check the validity of the results of the relativistic kinetic theory for a gas in the presence of a gravitational field. It seems of interest to check if both methods give the same results.

In section 1 we study a classical relativistic ideal gas in thermodynamic equilibrium in a uniformly accelerated reference frame using Gibbs's microcanonical and grand canonical formulations of statistical mechanics.

In section 2 we compare the results of section 1 with the well known results of the relativistic kinetic theory. We find complete agreement.

In section 3 we show that the expressions for the particle, energy and entropy density distributions can be integrated and we obtain explicit expressions for the total entropy, total energy and rest mass of the gas. We also find the center of mass of the system in equilibrium and derive the condition for the relativistic gas to be in mechanical equilibrium in the uniformly accelerated frame.

The non-relativistic approximation is considered in the first part of section 4. The approximate expressions for the total energy and total entropy are found to be in agreement with known results for a classical non-relativistic ideal gas in a uniform gravitational field.

The ultrarelativistic approximation is studied in the second part of section 4. Explicit expressions are found in this case using the microcanonical formalism. In particular, the phase space volume of the system can be found explicitly in this case, which allows a direct comparison between the results of the microcanonical and the grand canonical formulations.

Section 1: The microcanonical and grand canonical formulations

Let us consider a gas consisting of a very large number N of identical (indistinguishable) structureless particles. Assume the particles of the gas do not interact with each other, except for elastic collisions among themselves and with the walls of the container (an ideal gas). Let us assume the gas to be adiabatically isolated: any changes in the total internal energy can only be the result of a change in the external parameters. We assume the container is in uniformly accelerated motion along the z -direction with respect to an inertial reference frame K' , such that the components of the four-vector acceleration \ddot{x}_o^μ ($\mu = 0, 1, 2, 3$) of the container obey the condition:

$$\ddot{x}_o'^2 = \eta_{\mu\nu} \ddot{x}_o'^\mu \ddot{x}_o'^\nu = -\frac{g^2}{c^2} < 0, \quad (1)$$

where g is a constant, $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$ and c is the speed of light.

For simplicity, we assume the container to be a cylinder of base area A and height L . The axis of the cylinder is aligned along the z -direction.

In the local reference frame K of the accelerated observer, with respect to whom the box is at rest, the metric tensor can be written as [8, 9]¹:

$$g_{\mu\nu} = \text{diag} \left(\left(1 + \frac{gz}{c^2} \right)^2, -1, -1, -1 \right). \quad (2)$$

We assume the cylinder containing the gas is in a region within a distance $\frac{c^2}{g}$ of the observer. In other words, we assume the container is small enough and close enough to the origin of the reference frame K that the metric (2) is valid everywhere inside the container.

The Lagrangian for a system of N non-interacting identical relativistic particles in the non-inertial uniformly accelerated reference frame K can be written as:

$$L = -mc^2 \sum_{a=1}^N \left(\left(1 + \frac{gz_a}{c^2} \right)^2 - \frac{v_a^2}{c^2} \right)^{\frac{1}{2}}, \quad (3)$$

where v_a^i ($i = 1, 2, 3$) are the components of the velocity of the a th particle in K , $v_a^2 = v_{ax}^2 + v_{ay}^2 + v_{az}^2$, and m is the mass of a particle.

From the Lagrangian (3) we can find the components of the momentum of the a th particle:

$$p_{ai} = \frac{mv_a^i}{\left(\left(1 + \frac{gz_a}{c^2} \right)^2 - \frac{v_a^2}{c^2} \right)^{\frac{1}{2}}}, \quad (4)$$

The Hamiltonian of the system can therefore be written as:

$$H = mc^2 \sum_{a=1}^N \left(1 + \frac{gz_a}{c^2} \right) \left(1 + \frac{p_a^2}{m^2 c^2} \right)^{\frac{1}{2}}, \quad (5)$$

where $p_a^2 = p_{ax}^2 + p_{ay}^2 + p_{az}^2$.

Let us assume the gas has reached thermodynamic equilibrium. It is assumed that any infinitesimally small portion of the fluid contains a very large number of particles.

In the non-inertial reference frame K , where all the portions of the fluid are at rest, the total energy of the fluid can be written as:

$$E = A \int_0^L \epsilon(z) dz, \quad (6)$$

where $\epsilon(z)$ can be interpreted as the energy density of the fluid measured with respect to K .

In the instantaneous (freely falling) proper inertial frame K' , in which the fluid is also at rest, the total energy of the fluid can be written as follows:

¹In the non-relativistic limit this metric takes the form: $g_{\mu\nu} = \text{diag} \left(1 + \frac{2gz}{c^2}, -1, -1, -1 \right)$, which is the expression for the metric describing a uniform gravitational field in this approximation[8].

$$E' = A \int_0^L \epsilon'(z') dz' = A \int_0^L \epsilon'(z) dz, \quad (7)$$

where ϵ' is the proper energy density (energy per unit proper volume) measured with respect to K' . Notice that both reference frames K' and K are at rest with respect to the fluid and with respect to each other, and the metric tensor in K is of the form (2), therefore $dx' = dx$, $dy' = dy$ and $dz' = dz$. The dimensions of the container are the same in both frames of reference.

In K' the particles of the gas move in rectilinear uniform motion in between collisions (as free particles), while in K they are subject to the effect of non-inertial forces. The two quantities ϵ and ϵ' are related as follows:

$$\epsilon = \left(1 + \frac{gz}{c^2}\right) \epsilon'. \quad (8)$$

The energy density ϵ of the gas (measured with respect to K) takes into account the effect of the non-inertial forces on the particles, while the proper energy density ϵ' does not.

In the non-inertial reference frame K , where all portions of the fluid are at rest, the components of the energy-momentum tensor of the fluid can be expressed as follows:

$$T^{\mu\nu} = \text{diag} \left(\frac{\epsilon'}{\left(1 + \frac{gz}{c^2}\right)^2}, p', p', p' \right), \quad (9)$$

where p' is the pressure in the fluid (a local function of position in the fluid) measured with respect to the instantaneous inertial rest frame K' .

Equating the covariant derivative of the energy-momentum tensor to zero (law of energy-momentum conservation) we obtain the condition [10, 11] for the fluid to be in a state of mechanical equilibrium:

$$\frac{dp'}{dz} = -\frac{g}{c^2} \frac{(\epsilon' + p')}{\left(1 + \frac{gz}{c^2}\right)}. \quad (10)$$

In the microcanonical formalism, the number $\Omega(E, N, g, A, L) = D(E, N, g, A, L) \delta E$ of accessible microstates of the system (for a fixed number of particles N and fixed external parameters g , A and L , and energy in the range $(E, E + \delta E)$) can be found from the phase space volume integral:

$$\begin{aligned} \Phi(E, N, g, A, L) &= \int \dots \int dx_1 dy_1 dz_1 dp_{1x} dp_{1y} dp_{1z} \dots dx_N dy_N dz_N dp_{Nx} dp_{Ny} dp_{Nz} \\ &\sum_{a=1}^N \left(1 + \frac{gz_a}{c^2}\right) \left(1 + \frac{p_a^2}{m^2 c^2}\right)^{\frac{1}{2}} \leq \frac{E}{mc^2} \end{aligned} \quad (11)$$

as follows:

$$\Omega(E, N, g, A, L) = \frac{1}{N!h^{3N}} \left(\frac{\partial \Phi(E, N, g, A, L)}{\partial E} \right)_{N,g,A,L} \delta E. \quad (12)$$

The phase space volume integral (11) is the volume in phase space enclosed by the surface:

$$\sum_{a=1}^N \left(1 + \frac{gz_a}{c^2} \right) \left(1 + \frac{1}{m^2 c^2} (p_{ax}^2 + p_{ay}^2 + p_{az}^2) \right)^{\frac{1}{2}} = \frac{E}{mc^2}.$$

In (12), h is Planck's constant and the $N!$ accounts for the fact that the particles of the gas are indistinguishable.

From (11) we immediately find:

$$\begin{aligned} \Phi(E, N, g, A, L) &= (4\pi A)^N \int \dots \int p_1^2 \dots p_N^2 dz_1 \dots dz_N dp_1 \dots dp_N \\ &\quad \sum_{a=1}^N \left(1 + \frac{gz_a}{c^2} \right) \left(1 + \frac{p_a^2}{m^2 c^2} \right)^{\frac{1}{2}} \leq \frac{E}{mc^2}. \end{aligned} \quad (13)$$

Assuming that the number of particles N is very large ($N! \approx \sqrt{2\pi N} \left(\frac{N}{e}\right)^N$ (Stirling's approximation)), the entropy of the isolated system in thermodynamic equilibrium can be found from (12) and (13) as:

$$S = k \ln \Omega(E, N, g, A, L), \quad (14)$$

where k is Boltzmann's constant.

The quantities T and μ can be found from the partial derivatives of the entropy as follows:

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E} \right)_{N,g,A,L}, \quad (15)$$

$$\frac{\mu}{T} = - \left(\frac{\partial S}{\partial N} \right)_{E,g,A,L}. \quad (16)$$

Let us consider a layer of the fluid (of area A) at a height z . This is an open subsystem. For this portion of the gas we can write the grand canonical partition function:

$$\begin{aligned} \mathcal{Z} &= \sum_{N=0}^{\infty} e^{\frac{\mu N}{kT}} \frac{(Adz)^N}{h^{3N} N!} \int \dots \int e^{-\sum_{a=1}^N \frac{mc^2}{kT} \left(1 + \frac{gz}{c^2} \right) \left(1 + \frac{p_a^2}{m^2 c^2} \right)^{\frac{1}{2}}} d^3 p_1 \dots d^3 p_N \\ &= \sum_{N=0}^{\infty} e^{\frac{\mu N}{kT}} \frac{(Adz)^N}{h^{3N} N!} \left(4\pi \int_0^{\infty} e^{-\frac{mc^2}{kT} \left(1 + \frac{gz}{c^2} \right) \left(1 + \frac{p^2}{m^2 c^2} \right)^{\frac{1}{2}}} p^2 dp \right)^N \\ &= \sum_{N=0}^{\infty} \frac{1}{N!} \left(e^{\frac{\mu}{kT}} 4\pi Adz \left(\frac{mc}{h} \right)^3 \int_0^{\infty} e^{-a(z)(1+x^2)^{\frac{1}{2}}} x^2 dx \right)^N \\ &= \exp \left(e^{\frac{\mu}{kT}} 4\pi Adz \left(\frac{mc}{h} \right)^3 \int_0^{\infty} e^{-a(z)(1+x^2)^{\frac{1}{2}}} x^2 dx \right), \end{aligned} \quad (17)$$

where:

$$a(z) = \frac{mc^2}{kT} \left(1 + \frac{gz}{c^2} \right). \quad (18)$$

We notice that:

$$\int_0^\infty e^{-a(1+x^2)^{\frac{1}{2}}} x^2 dx = \int_1^\infty u \sqrt{u^2 - 1} e^{-au} du = \frac{K_2(a)}{a}, \quad (19)$$

where K_2 is the modified Bessel function [12, 13].

From (17), (18) and (19) we finally obtain the grand canonical partition function for the relativistic ideal gas:

$$\mathcal{Z} = \exp \left(e^{\frac{\mu}{kT}} 4\pi A dz \left(\frac{mc}{h} \right)^3 \frac{K_2 \left(\frac{mc^2}{kT} \left(1 + \frac{gz}{c^2} \right) \right)}{\frac{mc^2}{kT} \left(1 + \frac{gz}{c^2} \right)} \right). \quad (20)$$

The grand canonical potential per unit volume in K , can be found directly from (20):

$$\omega = -k e^{\frac{\mu}{kT}} 4\pi \left(\frac{mc}{h} \right)^3 \frac{K_2 \left(\frac{mc^2}{kT} \left(1 + \frac{gz}{c^2} \right) \right)}{\frac{mc^2}{kT} \left(1 + \frac{gz}{c^2} \right)}. \quad (21)$$

From (21) the entropy density s (entropy per unit volume) and the particle density n (number of particles per unit volume) can be found using the equations:

$$s = - \left(\frac{\partial \omega}{\partial T} \right)_{g, \mu}, \quad (22)$$

$$n = - \left(\frac{\partial \omega}{\partial \mu} \right)_{g, T}. \quad (23)$$

Using the thermodynamic identity:

$$\omega = \epsilon - Ts - \mu n, \quad (24)$$

and expressing μ in terms of the particle density $n(0)$ at the bottom of the container, we obtain from (21), (22), (23) and (24) the particle density, entropy density, and energy density equilibrium distributions:

$$n(z) = n(0) \frac{K_2 \left(\frac{mc^2}{kT} \left(1 + \frac{gz}{c^2} \right) \right)}{\left(1 + \frac{gz}{c^2} \right) K_2 \left(\frac{mc^2}{kT} \right)}, \quad (25)$$

$$s(z) = n(z) k \left[\ln \left(\frac{4\pi e^2}{n(z)} \left(\frac{mc}{h} \right)^3 \frac{K_2 \left(\frac{mc^2}{kT} \left(1 + \frac{gz}{c^2} \right) \right)}{\frac{mc^2}{kT} \left(1 + \frac{gz}{c^2} \right)} \right) - \frac{mc^2}{kT} \left(1 + \frac{gz}{c^2} \right) \frac{K_2' \left(\frac{mc^2}{kT} \left(1 + \frac{gz}{c^2} \right) \right)}{K_2 \left(\frac{mc^2}{kT} \left(1 + \frac{gz}{c^2} \right) \right)} \right], \quad (26)$$

$$\epsilon(z) = n(z)kT \left(1 - \frac{mc^2}{kT} \left(1 + \frac{gz}{c^2} \right) \frac{K_2' \left(\frac{mc^2}{kT} \left(1 + \frac{gz}{c^2} \right) \right)}{K_2 \left(\frac{mc^2}{kT} \left(1 + \frac{gz}{c^2} \right) \right)} \right), \quad (27)$$

where $0 \leq z \leq L$ (we assume the origin of the reference frame K to be at the bottom of the container).

The pressure in K can be found directly from the expression of the grand canonical potential per unit volume as follows:

$$p = -\omega = n(z)kT \quad (28)$$

From (27) and (8) we find the proper energy density as a function of z :

$$\epsilon'(z) = \frac{n(z)kT}{\left(1 + \frac{gz}{c^2}\right)} \left(1 - \frac{mc^2}{kT} \left(1 + \frac{gz}{c^2} \right) \frac{K_2' \left(\frac{mc^2}{kT} \left(1 + \frac{gz}{c^2} \right) \right)}{K_2 \left(\frac{mc^2}{kT} \left(1 + \frac{gz}{c^2} \right) \right)} \right). \quad (29)$$

Substituting (29) into (10), we can solve the differential equation for the pressure p' in K' to obtain:

$$p'(z) = \frac{n(z)kT}{\left(1 + \frac{gz}{c^2}\right)}. \quad (30)$$

The thermodynamic relations between ϵ' , n' , s' , p' and the local temperature T' measured in the instantaneous inertial frame K' for a layer of gas in equilibrium at height z can be found from the grand canonical partition function (in the inertial frame):

$$\begin{aligned} \mathcal{Z}' &= \sum_{N=0}^{\infty} e^{\frac{\mu'(z)N}{kT'(z)}} \frac{(Adz)^N}{h^{3N}N!} \int \dots \int e^{-\sum_{a=1}^N \frac{mc^2}{kT'(z)} \left(1 + \frac{p_a'^2}{m^2 c^2} \right)^{\frac{1}{2}}} d^3 p'_1 \dots d^3 p'_N \\ &= \exp \left(e^{\frac{\mu'(z)}{kT'(z)}} 4\pi Adz \left(\frac{mc}{h} \right)^3 \frac{K_2 \left(\frac{mc^2}{kT'(z)} \right)}{\frac{mc^2}{kT'(z)}} \right). \end{aligned} \quad (31)$$

As before, we can find s' and n' from the equations:

$$s' = - \left(\frac{\partial \omega'}{\partial T'} \right)_{\mu'} , \quad (32)$$

$$n' = - \left(\frac{\partial \omega'}{\partial \mu'} \right)_{T'} . \quad (33)$$

where ω' can be found from (31):

$$\omega' = -k e^{\frac{\mu'}{kT'}} 4\pi \left(\frac{mc}{h} \right)^3 \frac{K_2 \left(\frac{mc^2}{kT'} \right)}{\frac{mc^2}{kT'}}. \quad (34)$$

Finally, using the thermodynamic identity:

$$\omega' = \epsilon' - T's' - \mu'n', \quad (35)$$

and the relation:

$$\omega' = -p', \quad (36)$$

for a relativistic ideal gas (in the inertial frame K'), in the absence of external forces, one finds the Juttner relations [14, 15, 16, 17, 18, 19, 20]:

$$\epsilon' = p' \left(1 - \frac{mc^2}{kT'} \frac{K'_2\left(\frac{mc^2}{kT'}\right)}{K_2\left(\frac{mc^2}{kT'}\right)} \right). \quad (37)$$

$$p' = n'kT' \quad (38)$$

$$s' = n'k \left[\ln \left(\frac{4\pi e^2}{n'} \left(\frac{mc}{h} \right)^3 \frac{K_2\left(\frac{mc^2}{kT'}\right)}{\frac{mc^2}{kT'}} \right) - \frac{mc^2}{kT'} \frac{K'_2\left(\frac{mc^2}{kT'}\right)}{K_2\left(\frac{mc^2}{kT'}\right)} \right], \quad (39)$$

Comparing (37), (38) and (39) with (29), (30) and (26) we obtain:

$$T'(z) = \frac{T}{\left(1 + \frac{gz}{c^2}\right)}, \quad (40)$$

$$n' = n, \quad (41)$$

$$s' = s. \quad (42)$$

Equations (41) and (42) can also be obtained by noticing that the number of particles and the entropy are invariant quantities (have the same values both in K and K') and the volumes in K and K' are the same (as noted above).

Equation (40) is the well known Tolman relation [21, 9] ($T'\sqrt{g_{00}} = T = \text{const}$) between the local temperature $T'(z)$ measured with respect to the inertial frame K' and the constant T defined in the non-inertial frame K . We have arrived at (40) using the methods of statistical mechanics.

Section 2: The relativistic kinetic theory

One can also obtain Eqs. (25 - 27) from the kinetic theory, using the relativistic Boltzmann equation [1, 2, 3, 4, 5, 6, 7].

In the non-inertial uniformly accelerated reference frame K , the relativistic Boltzmann equation can be written as follows:

$$\begin{aligned}
& \frac{\partial f(t, \vec{r}, \vec{p})}{\partial t} + \frac{\left(1 + \frac{gz}{c^2}\right)}{m \left(1 + \frac{p^2}{m^2 c^2}\right)^{\frac{1}{2}}} \left(\vec{p} \frac{\partial f(t, \vec{r}, \vec{p})}{\partial \vec{r}} \right) - mg \left(1 + \frac{p^2}{m^2 c^2}\right)^{\frac{1}{2}} \frac{\partial f(t, \vec{r}, \vec{p})}{\partial p_z} \\
&= \frac{1}{h^3} \int \omega(\vec{r}; \vec{p}, \vec{p}_1; \vec{\tilde{p}}, \vec{\tilde{p}}_1) \left(f(t, \vec{r}, \vec{\tilde{p}}) f(t, \vec{r}, \vec{\tilde{p}}_1) - f(t, \vec{r}, \vec{p}) f(t, \vec{r}, \vec{p}_1) \right) d^3 p_1 d^3 \tilde{p} d^3 \tilde{p}_1.
\end{aligned} \tag{43}$$

The right hand side of the relativistic Boltzmann equation (43) is the collision integral. It gives the rate of change, due to collisions, in the mean number of particles in the phase space volume $dx dy dz dp_x dp_y dp_z$. The left hand side of (43) is $\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\}$, where $\{, \}$ is a Poisson bracket. The particular expression in the left hand side of (43) is obtained by inserting the Hamiltonian (5) in the Poisson bracket.

For elastic collisions:

$$\vec{p} + \vec{p}_1 = \vec{\tilde{p}} + \vec{\tilde{p}}_1, \tag{44}$$

$$\left(1 + \frac{p^2}{m^2 c^2}\right)^{\frac{1}{2}} + \left(1 + \frac{p_1^2}{m^2 c^2}\right)^{\frac{1}{2}} = \left(1 + \frac{\tilde{p}^2}{m^2 c^2}\right)^{\frac{1}{2}} + \left(1 + \frac{\tilde{p}_1^2}{m^2 c^2}\right)^{\frac{1}{2}}. \tag{45}$$

The mean number of particles per unit volume is given by the equation:

$$n(t, \vec{r}) = \frac{1}{h^3} \int f(t, \vec{r}, \vec{p}) d^3 p. \tag{46}$$

In equilibrium:

$$\frac{\partial f_{eq}}{\partial t} = 0, \tag{47}$$

$$f_{eq}(\vec{r}, \vec{\tilde{p}}) f_{eq}(\vec{r}, \vec{\tilde{p}}_1) = f_{eq}(\vec{r}, \vec{p}) f_{eq}(\vec{r}, \vec{p}_1). \tag{48}$$

In Eq (48) it is assumed that the initial and final momenta obey the conditions (44, 45).

In our case, due to the symmetry of the problem considered, we also have the following conditions on f_{eq} :

$$\frac{\partial f_{eq}}{\partial x} = \frac{\partial f_{eq}}{\partial y} = 0. \tag{49}$$

From (43), (47 - 49) we obtain that the distribution function f_{eq} satisfies the equation:

$$\frac{\left(1 + \frac{gz}{c^2}\right) p_z}{m \left(1 + \frac{p^2}{m^2 c^2}\right)^{\frac{1}{2}}} \frac{\partial f_{eq}(z, \vec{p})}{\partial z} - mg \left(1 + \frac{p^2}{m^2 c^2}\right)^{\frac{1}{2}} \frac{\partial f_{eq}(z, \vec{p})}{\partial p_z} = 0. \tag{50}$$

The solution of (50), which also satisfies the condition (48) (assuming (44, 45)), can be written as:

$$f_{eq}(z, \vec{p}) = \alpha e^{-\beta mc^2 \left(1 + \frac{gz}{c^2}\right) \left(1 + \frac{p^2}{m^2 c^2}\right)^{\frac{1}{2}}}, \quad (51)$$

where α and β are constants.

Substituting (51) into (46) and performing the integration we obtain:

$$n(z) = \frac{4\pi\alpha m^2 c}{\beta} \frac{K_2\left(\beta mc^2 \left(1 + \frac{gz}{c^2}\right)\right)}{\left(1 + \frac{gz}{c^2}\right)}. \quad (52)$$

In terms of the particle density at the bottom of the container we obtain:

$$n(z) = \frac{n(0)}{K_2(\beta mc^2)} \frac{K_2\left(\beta mc^2 \left(1 + \frac{gz}{c^2}\right)\right)}{\left(1 + \frac{gz}{c^2}\right)}. \quad (53)$$

The energy density $\epsilon(z)$ in K can be derived from the relation:

$$\epsilon(z) = \frac{1}{h^3} \int mc^2 \left(1 + \frac{gz}{c^2}\right) \left(1 + \frac{p^2}{m^2 c^2}\right)^{\frac{1}{2}} f_{eq}(z, \vec{p}) d^3 p \quad (54)$$

Substituting (51) into (54), taking into account (52), and performing the integration we obtain:

$$\epsilon(z) = \frac{n(z)}{\beta} \left(1 - \beta mc^2 \left(1 + \frac{gz}{c^2}\right) \frac{K'_2\left(\beta mc^2 \left(1 + \frac{gz}{c^2}\right)\right)}{K_2\left(\beta mc^2 \left(1 + \frac{gz}{c^2}\right)\right)}\right) \quad (55)$$

Finally, we can obtain the entropy density from the relation [22]:

$$s(z) = \frac{k}{h^3} \int f_{eq} \ln\left(\frac{e}{f_{eq}}\right) d^3 p. \quad (56)$$

Substituting (51) into (56), taking into account (52), and performing the integration we obtain:

$$s(z) = n(z)k \left[\ln\left(\frac{4\pi e^2}{n(z)} \left(\frac{mc}{h}\right)^3 \frac{K_2\left(\beta mc^2 \left(1 + \frac{gz}{c^2}\right)\right)}{\beta mc^2 \left(1 + \frac{gz}{c^2}\right)}\right) - \beta mc^2 \left(1 + \frac{gz}{c^2}\right) \frac{K'_2\left(\beta mc^2 \left(1 + \frac{gz}{c^2}\right)\right)}{K_2\left(\beta mc^2 \left(1 + \frac{gz}{c^2}\right)\right)} \right], \quad (57)$$

Comparing (53, 54, 57) with (25 - 27) we find the constant β :

$$\beta = \frac{1}{kT}. \quad (58)$$

Section 3: Global quantities

The density of particles at the bottom of the container $n(0)$ can be related to the total number of particles in the system N by direct integration of (25):

$$N = A \int_0^L n(z) dz = \frac{An(0)}{K_2\left(\frac{mc^2}{kT}\right)} \frac{c^2}{g} \int_{a(0)}^{a(L)} \frac{K_2(u)}{u} du = \frac{An(0)}{K_2\left(\frac{mc^2}{kT}\right)} \frac{kT}{mg} \left(K_1\left(\frac{mc^2}{kT}\right) - \frac{K_1\left(\frac{mc^2}{kT}\left(1 + \frac{gL}{c^2}\right)\right)}{\left(1 + \frac{gL}{c^2}\right)} \right), \quad (59)$$

where we used the relation: $\frac{K_2(u)}{u} = -\frac{d}{du} \left(\frac{K_1(u)}{u} \right)$ [12, 13].

The total energy of the system E in K can be found by substituting (25) and (59) into (27) and performing the integration (6):

$$E = NkT + Nmc^2 \frac{\left(K_2\left(\frac{mc^2}{kT}\right) - K_2\left(\frac{mc^2}{kT}\left(1 + \frac{gL}{c^2}\right)\right) \right)}{\left(K_1\left(\frac{mc^2}{kT}\right) - \frac{K_1\left(\frac{mc^2}{kT}\left(1 + \frac{gL}{c^2}\right)\right)}{\left(1 + \frac{gL}{c^2}\right)} \right)}. \quad (60)$$

Similarly, substituting (25) and (59) into (26), and integrating, we obtain the total entropy of the gas:

$$S = Nk \ln \left(\frac{A}{N} 4\pi e^2 \left(\frac{mc}{h} \right)^3 \left(\frac{kT}{mc^2} \right)^2 \frac{c^2}{g} \left(K_1\left(\frac{mc^2}{kT}\right) - \frac{K_1\left(\frac{mc^2}{kT}\left(1 + \frac{gL}{c^2}\right)\right)}{\left(1 + \frac{gL}{c^2}\right)} \right) \right) + \frac{Nmc^2}{T} \frac{\left(K_2\left(\frac{mc^2}{kT}\right) - K_2\left(\frac{mc^2}{kT}\left(1 + \frac{gL}{c^2}\right)\right) \right)}{\left(K_1\left(\frac{mc^2}{kT}\right) - \frac{K_1\left(\frac{mc^2}{kT}\left(1 + \frac{gL}{c^2}\right)\right)}{\left(1 + \frac{gL}{c^2}\right)} \right)}. \quad (61)$$

The total energy of the system E' in K' can be found by substituting (25) and (59) into (29) and performing the integration (7):

$$E' = Nmc^2 \frac{\left(K_2\left(\frac{mc^2}{kT}\right) - \frac{K_2\left(\frac{mc^2}{kT}\left(1 + \frac{gL}{c^2}\right)\right)}{\left(1 + \frac{gL}{c^2}\right)} \right)}{\left(K_1\left(\frac{mc^2}{kT}\right) - \frac{K_1\left(\frac{mc^2}{kT}\left(1 + \frac{gL}{c^2}\right)\right)}{\left(1 + \frac{gL}{c^2}\right)} \right)}. \quad (62)$$

E' given by Eq(62) is the proper energy of the gas.

The position of the center of mass of the gas with respect to the inertial frame K' can be determined by the equation [23]:

$$z_{c.m.} = \frac{A \int_0^L z \epsilon'(z) dz}{E'} \quad (63)$$

The integration in (63) can be easily carried out and using (62) and (60) one finds that:

$$Mgz_{c.m.} = E - E'. \quad (64)$$

where the rest mass M of the fluid is defined as:

$$M = \frac{E'}{c^2} \quad (65)$$

From (65), (62), (28), (25) and (59), we notice that:

$$Mg = A(p(0) - p(L)). \quad (66)$$

The above equation can be interpreted as the condition (in K) for the object to be in static mechanical equilibrium: the net force exerted by the walls of the container on the gas as a whole equals its weight. Notice that in the non-inertial reference frame K the net force exerted on the gas by the side walls of the cylinder is equal to zero.

Section 4: The non-relativistic and ultrarelativistic approximations

The non-relativistic approximation.

In the non-relativistic limit ($\frac{mc^2}{kT} \rightarrow \infty$) we can use the expressions for the asymptotics of the modified Bessel functions [12]:

$$K_\nu(z) \approx \sqrt{\frac{\pi}{2z}} e^{-z} \left(1 + \frac{4\nu^2 - 1}{8z} \right), \quad (67)$$

for $\nu = 1, 2$ in (60) and (61) to obtain the non-relativistic expressions for the total energy and total entropy of the gas:

$$E = Nmc^2 + \frac{5}{2}NkT - \frac{NmgL}{\left(e^{\frac{mgL}{kT}} - 1\right)}, \quad (68)$$

$$S = Nk \left(1 - \frac{mgL}{kT} \frac{1}{\left(e^{\frac{mgL}{kT}} - 1\right)} \right) + Nk \ln \left(\frac{A}{N} \frac{(2\pi mkT)^{\frac{3}{2}}}{h^3} e^{\frac{5}{2}} \frac{kT}{mg} \left(1 - e^{-\frac{mgL}{kT}} \right) \right). \quad (69)$$

Equations (68) and (69) are in perfect agreement with the results for the non-relativistic classical ideal gas in a uniform gravitational field obtained in [24].

In the non-relativistic limit, where $\left(1 + \frac{p_a^2}{m^2 c^2}\right)^{\frac{1}{2}} \approx 1 + \frac{p_a^2}{2m^2 c^2}$, the integration in (13) can also be carried out and (assuming $E > Nmc^2 + NmgL$) we obtain results in agreement with [25].

The ultrarelativistic approximation.

In the ultrarelativistic approximation, where $\left(1 + \frac{p_a^2}{m^2 c^2}\right)^{\frac{1}{2}} \approx \frac{p_a}{mc}$, the integration in (13) can be carried out explicitly and (assuming $E > Nmc^2 \left(1 + \frac{gL}{c^2}\right)$) we obtain:

$$\Phi(E, N, g, A, L) = \frac{1}{(3N)!} \left(\frac{4\pi A}{gc} E^3 \left(1 - \frac{1}{\left(1 + \frac{gL}{c^2}\right)^2} \right) \right)^N. \quad (70)$$

From (70), (12) and (14), using Stirling's approximation, we obtain a formula for the entropy of an ultrarelativistic ideal gas in thermodynamic equilibrium in a uniformly accelerated reference frame:

$$S = Nk \ln \left(\frac{4\pi e^4}{gch^3} \frac{A}{N} \left(\frac{E}{3N} \right)^3 \left(1 - \frac{1}{\left(1 + \frac{gL}{c^2} \right)^2} \right) \right). \quad (71)$$

In the ultrarelativistic limit ($\frac{mc^2}{kT} \rightarrow 0$) we can use the approximate expressions [12, 13] for the modified Bessel functions as $z \rightarrow 0$:

$$K_\nu(z) \approx \frac{\Gamma(\nu)}{2} \left(\frac{2}{z} \right)^\nu \quad (72)$$

for $\nu = 1, 2$, in (60) and (61) to obtain:

$$E = 3NkT \quad (73)$$

$$S = Nk \ln \left(\frac{4\pi e^4}{gch^3} \frac{A}{N} (kT)^3 \left(1 - \frac{1}{\left(1 + \frac{gL}{c^2} \right)^2} \right) \right). \quad (74)$$

Combining Eqs. (74) and (73) we obtain (71), which shows that the microcanonical and the grand canonical formalisms give identical results in the ultrarelativistic approximation.

In this paper we have considered the case of an ideal gas in a uniformly accelerated frame. The relativistic kinetic theory is applicable in the more general case of a gas in a stationary gravitational field. It would be of interest to generalize the results obtained here to find the microcanonical and grand canonical formulations for the gas in a stationary gravitational field.

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